ISRAEL JOURNAL OF MATHEMATICS **128** (2002), 325–354

SPECTRA OF COMPOSITION OPERATORS ON THE BLOCH AND BERGMAN SPACES

BY

BARBARA MACCLUER

Department of Mathematics, Kerchof Hall
University of Virginia, Charlottesville, VA 22904, USA
email: bdm3f@virginia.edu

AND

KAREN SAXE

Department of Mathematics, Macalester College
1600 Grand Avenue, St. Paul, MN 55105, USA
email: saxe@macalester.edu

ABSTRACT

When φ is an analytic map of the unit disk U into itself, and X is a Banach space of analytic functions on U, define the composition operator C_φ by $C_\varphi(f) = f\circ\varphi$, for $f \in X$. In this paper we show how to use the Calderón theory of complex interpolation to obtain information on the spectrum of C_φ (under suitable hypotheses on φ) acting on the Bloch space $\mathcal{B}$ and BMOA, the space of analytic functions in BMO. To do this we first obtain some results on the essential spectral radius and spectrum of C_φ on the Bergman spaces A^p and Hardy spaces H^p, spaces which are connected to $\mathcal{B}$ and BMOA by the interpolation relationships $[A^1, \mathcal{B}]_t = A^p$ and $[H^1, BMOA]_t = H^p$ for $1 = p(1-t)$.

1. Introduction

In this paper we determine the spectra of some composition operators on the Bergman spaces, and on the Hardy spaces. We then use this information, apply Calderón interpolation methods, and glean information about the spectrum on the Bloch space and BMOA.

Received July 24, 2000

Throughout the entire paper, the open unit disk in the complex plane will be denoted by U, φ will denote an analytic map from U to itself, φ_n its nth iterate, and C_φ the associated composition operator given by

$$C_\varphi(f) = f \circ \varphi.$$

All spaces that we consider are collections of functions analytic on the disk U. For $1 \leq p < \infty$, $H^p = H^p(U)$ is the Hardy space of functions f that are analytic on U and satisfy

$$\|f\|_{H^p}^p = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

and $A^p = A^p(U)$ is the Bergman space of functions f that are analytic on U and satisfy

$$\|f\|_{A^p}^p = \int_U |f(z)|^p dA(z) < \infty,$$

where dA is normalized Lebesgue measure on U. A function f is in BMOA if and only if $f \in H^2$ and

$$\|f\|_* = \sup\{\|f \circ \varphi_a - f(a)\|_{H^2} : a \in U\} < \infty,$$

where

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

The space BMOA is normed by

$$\|f\|_{BMOA} = |f(0)| + \|f\|_*.$$

This is equivalent to the more traditional definition of BMOA by the John–Nirenberg theorem. The Bloch space consists of the analytic functions on U that satisfy

$$M(f) = \sup\{(1 - |z|^2)|f'(z)| : z \in U\} < \infty,$$

and is normed by

$$\|f\|_{\mathcal{B}} = |f(0)| + M(f).$$

Each of the spaces defined above is a Banach space.

The formula $C_\varphi(f) = f \circ \varphi$ defines a bounded linear operator on each of these Banach spaces. Boundedness on the Hardy and Bergman spaces is by now considered standard (see, for example, [9]). The first proof we can find of boundedness on BMOA and the Bloch space appears in Theorem 2 of [1]. See also [5] and [21] (for BMOA), and [13] (for Bloch).

The operator norm, spectral radius, and spectrum of a bounded linear operator T, when regarded as an operator on the Banach space X, will be denoted by $\|T\|_X$, $r_X(T)$, and $\sigma_X(T)$, respectively. The essential norm, essential spectral radius, and essential spectrum of T, when regarded as an operator on the Banach space X, will be denoted by $\|T\|_{e,X}$, $r_{e,X}(T)$, and $\sigma_{e,X}(T)$, respectively. These notations occasionally will be abbreviated when the space X is clear from context. The adjective "essential" indicates that T is being considered as an element of the Calkin algebra (the Banach algebra of bounded linear operators modulo the compact operators).

In [6], Paul Bourdon and Joel Shapiro give the intriguing essential spectral radius formula

$$\left(r_{e,H^p}(C_\varphi)\right)^p = \left(r_{e,H^2}(C_\varphi)\right)^2$$

for any analytic $\varphi: U \to U$. Their proof of the inequality

$$\left(r_{e,H^p}(C_\varphi)\right)^p \leq \left(r_{e,H^2}(C_\varphi)\right)^2$$

holds for any value of $p, 0 < p < \infty$ (when $0 < p < 1$ the spaces are p-Banach spaces). However, their proof for the opposite inequality, while stated to hold for all $0 < p < \infty$, does not seem complete in the range $0 < p \leq 1$. (The assertion that if f_n are unit vectors in H^p tending to 0 uniformly on compact subsets of U and K is any compact operator on H^p, then $\|Kf_n\|_{H^p} \to 0$, is not correct. We thank the authors for their correspondence with us on this issue.) Our first main result obtains a Bergman space analogue of this equality on the essential spectral radius, for $p > 1$. Our second main result gives the spectrum, $\sigma_{A^p}(C_\varphi)$, in terms of $r_{e,A^p}(C_\varphi)$. Specifically,

$$\sigma_{A^p}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,A^p}(C_\varphi)\} \cup \{(\varphi'(a))^n\}_{n=0}^{\infty},$$

for φ univalent, not an automorphism, with fixed point $a \in U$, and $p \geq 1$. This extends Corollaries 19 and 24 of [8], where $\sigma_{H^2}(C_\varphi)$ and $\sigma_{A^2}(C_\varphi)$ are given by formulas of the same type. Along these same lines, we point out that Lixin Zheng in [24] has shown that $\sigma_{H^\infty}(C_\varphi) = \overline{U}$, when φ, not an automorphism, fixes a point of U. Our third main result uses our first two results and goes part of the way in determining $\sigma_{\mathcal{B}}(C_\varphi)$, for φ not an automorphism, fixing a point in U and univalent. We will also show that our main results still hold when A^p is replaced by H^p and the Bloch space by BMOA.

The rest of the paper is organized as follows. The next section contains preliminary material on three topics: basic facts about the Nevanlinna counting function and its role in change-of-variable formulas, norms of certain evaluation functionals on A^p and some of its subspaces, and a summary of results we need about the Calderón method of complex interpolation. Section 3 contains the main spectral results on Bergman and Bloch spaces, beginning with a Bergman space analog of Bourdon and Shapiro's essential spectral radius relationship and culminating in our description of the spectrum of C_φ on A^p for $p \geq 1$ (with certain conditions on φ) and a partial description of the spectrum on $\mathcal{B}$. Section 4 develops the analogous spectral results for H^p for $p \geq 1$, and BMOA. The final section gives some examples and discusses a conjecture for the spectrum of C_φ on $\mathcal{B}$ and BMOA.

2. Preliminary material

2.1 The Generalized Nevanlinna Counting Function. In proving the A^p version of Bourdon and Shapiro's essential spectral radius formula, we make use of the generalized Nevanlinna counting function

$$N_{\varphi,\gamma}(z) = \sum_{w \in \{\varphi^{-1}(z)\}} \left(\log \frac{1}{|w|}\right)^\gamma, \quad \gamma > 0, \quad z \in U \setminus \{\varphi(0)\}.$$

The sum is taken over the preimages of z, counting multiplicities, and when $z \notin \varphi(U)$, $N_{\varphi,\gamma}(z)$ is defined to be 0. We will need to make use of several results involving this function. These are now listed.

Change of variable formula (Formula 6.4 of [19]): *If f is a positive measurable function on U and φ is an analytic function mapping U to itself, then*

$$\int_U f(\varphi(z))|\varphi'(z)|^2 \left(\log \frac{1}{|z|}\right)^\gamma dA(z) = \int_U f(z) N_{\varphi,\gamma}(z) dA(z).$$

This result gives rise to two formulas which we will use. They tell us about the operator norm of the composition operator C_φ on H^p and A^p, respectively. These formulas hold for $0 < p < \infty$ (see [20]):

$$\|f \circ \varphi\|_{H^p}^p = |f(\varphi(0))|^p + \frac{p^2}{2} \int_U |f(z)|^{p-2} |f'(z)|^2 N_{\varphi,1}(z) dA(z). \tag{1.1}$$

$$\|f \circ \varphi\|_{A^p}^p \approx |f(\varphi(0))|^p + \int_U |f(z)|^{p-2} |f'(z)|^2 N_{\varphi,2}(z) dA(z). \tag{1.2}$$

Formula (1.2) is a special case of Proposition 2.4 of [20].

The symbol "$\approx$" means that the left hand side is bounded below and above by positive constant multiples of the right hand side; the constants do not depend on f.

If we apply equation (1.1) to the function $\varphi(z) = rz$ we see that, for f fixing the origin,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \frac{p^2}{2}\int_{rU} |f(z)|^{p-2}|f'(z)|^2 \log\frac{r}{|z|} dA(z). \tag{1.3}$$

We will also make use of formulas for the essential norm of a composition operator on H^2 and A^2, which are given in terms of the Nevanlinna counting functions. These are due to Shapiro (Theorem 2.3 of [19]) and Pietro Poggi-Corradini (Theorem 1.1 of [16]), respectively.

$$\|C_\varphi\|^2_{e,H^2} = \limsup_{|a|\to 1^-} \frac{N_{\varphi,1}(a)}{\log(\frac{1}{|a|})}. \tag{1.4}$$

$$\|C_\varphi\|^2_{e,A^2} = \limsup_{|a|\to 1^-} \frac{N_{\varphi,2}(a)}{(\log(\frac{1}{|a|}))^2}. \tag{1.5}$$

Formula (1.5) is a special case of Theorem 1.1 of [16].

2.2 Certain Invariant Subspaces for Composition Operators. Let m be a positive integer and suppose that $\varphi\colon U \to U$ fixes the origin. The idea of considering C_φ on the subspace $A_m \equiv z^m A^p$ of A^p will appear many times in our proofs. Note that A_m is equivalently described as

$$\{g \in A^p : g \text{ has a zero of at least order } m \text{ at zero}\}.$$

Because $\varphi(0) = 0$, A_m is an invariant subspace for C_φ. Since A_m is an invariant subspace with finite codimension in A^p, any operator that is invertible on A^p must also be invertible on A_m. This follows similarly to Lemma 7.17 in [9] after noting that a modification of the same proof works for Banach spaces (it appears in [9] for Hilbert spaces only). We will use $\sigma_{A_m}(C_\varphi) \subseteq \sigma_{A^p}(C_\varphi)$, which is an immediate consequence of this fact.

2.3 Evaluation Functionals. The next several results concern the norm of the linear functionals of evaluation at $w \in U$, or evaluation of the first derivative at $w \in U$, on the Banach space A^p and on the invariant subspaces A_m discussed in the previous subsection. Throughout the rest of the paper, ev_w will denote the linear functional of evaluation at $w \in U$, that is $ev_w(f) \equiv f(w)$ for f in some Banach space of analytic functions. We emphasize that the space on which ev_w acts will change from time to time, but no change in the notation for the functional will be made to indicate this.

The first result is known. We restrict attention to $p \geq 1$, although with a suitable interpretation the result extends to $0 < p < 1$.

PROPOSITION 1: (a) *Consider ev_w, for $w \in U$, acting on A^p, $1 \le p < \infty$. Then*

$$\|ev_w\|_{A^p} = \frac{1}{(1-|w|^2)^{2/p}}.$$

(b) *For $1 \le p < \infty$ there exists a constant $c(p)$, depending only on p, so that for any $w \in U$ and $f \in A^p$,*

$$|f'(w)| \le c(p)(1-|w|)^{-(p+2)/p}\|f\|_{A^p}.$$

A proof of (a) can be found in [22]. For (b) use Cauchy's formula

$$f'(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}dz,$$

where γ is the circle centered at w with radius $(1-|w|)/2$, and the upper estimate from (a) for $|f(z)|$ when $z \in \gamma$.

The content of the next result is an estimate for the norm of the evaluation functional acting on the subspaces A_m of A^p.

PROPOSITION 2: *Let $1 \le p < \infty$. Then there is a constant, $c(p)$, depending only on p, so that if $f \in A_m$ and $w \in U$,*

$$|f(w)| \le c(p)\ \frac{2m+1}{(1-|w|^2)^{2/p}}|w|^m\|f\|_{A^p}.$$

Proof: It suffices to prove the result for f a polynomial in A_m, since these polynomials form a dense subset of A_m. Since the polynomials are in A^2, we can calculate $|f(w)|$ as $|(f, K_w^m)|$ where the inner product is the usual A^2 inner product and K_w^m denotes the kernel function for evaluation at w in $z^m A^2$.

The kernel function is

$$K_w^m(z) = \sum_{k=m}^{\infty}(1+k)(\overline{w}z)^k = \sum_{k=m+1}^{\infty} k(\overline{w}z)^{k-1}.$$

Differentiating the geometric series formula for $\sum_{k=m+1}^{\infty} t^k$ we obtain the expression

$$K_w^m(z) = (\overline{w}z)^m\Big[\frac{(m+1)-(\overline{w}z)m}{(1-\overline{w}z)^2}\Big].$$

Thus, if $1 < p < \infty$ and $1/p + 1/q = 1$,

$$\begin{aligned}
\|K_w^m\|_{A^q} &= \Big[\int_U |K_w^m(z)|^q dA(z)\Big]^{1/q} \\
&= \Big[\int_U |\overline{w}z|^{mq}\Big|\frac{(m+1)-(\overline{w}z)m}{(1-\overline{w}z)^2}\Big|^q dA(z)\Big]^{1/q} \\
&\leq |w|^m\Big[\int_U \frac{((m+1)+m)^q}{|1-\overline{w}z|^{2q}} dA(z)\Big]^{1/q} \\
&\leq |w|^m(2m+1)\Big[\int_U \frac{1}{|1-\overline{w}z|^{2q}} dA(z)\Big]^{1/q}.
\end{aligned}$$

By Lemma 3.10 of [2], the last integral in this string of inequalities is bounded below and above by multiples (independent of w) of

$$(1-|w|^2)^{(2-2q)/q} = (1-|w|^2)^{-2/p}.$$

Now suppose that $f \in A_m$ is a polynomial. When $p > 1$, we apply Hölder's inequality and get

$$\begin{aligned}
|f(w)| = |(f, K_w^m)| &= \Big|\int_U f(z)\overline{K_w^m(z)} dA(z)\Big| \\
&\leq \Big(\int_U |f(z)|^p dA(z)\Big)^{1/p}\Big(\int_U |K_w^m(z)|^q dA(z)\Big)^{1/q} \\
&\leq c(p)(2m+1)|w|^m \frac{1}{(1-|w|^2)^{2/p}} \|f\|_{A^p},
\end{aligned}$$

as desired. Since

$$\|K_w^m\|_\infty \leq |w|^m \frac{2m+1}{(1-|w|)^2}$$

a similar calculation gives the result for $p = 1$. ∎

PROPOSITION 3: *Let $1 \leq p < \infty$. For every $w \in U$ with $|w| \geq \frac{1}{2}$,*

$$\|ev_w\|_{A_m} \leq \|ev_w\|_{A^p} \leq 2^m \|ev_w\|_{A_m}.$$

Proof: The inequality $\|ev_w\|_{A_m} \leq \|ev_w\|_{A^p}$ is obvious. For the other inequality, we first use the functions

$$f_w(z) = z^m\Big(\frac{1-|w|^2}{(1-\overline{w}z)^2}\Big)^{2/p}$$

which are in the unit ball of A_m to see that

$$\|ev_w\|_{A_m} \geq \frac{|w|^m}{(1-|w|^2)^{2/p}}.$$

Using this and Proposition 1 we have

$$\|ev_w\|_{A^p} = \frac{1}{(1-|w|^2)^{2/p}} \leq 2^m \|ev_w\|_{A_m}$$

whenever $|w| \geq \frac{1}{2}$. ∎

Part (b) of Proposition 1 gives an estimate for $|f'(w)|$ when $f \in A^p$. The next result gives an estimate for $|f'(w)|$ when $f \in z^m A^2$.

PROPOSITION 4: *Suppose $f \in z^m A^2$. Then for every $z \in U$,*

$$|f'(w)| \leq |w|^{m-1} \frac{2\sqrt{3}m^{3/2}}{(1-|w|^2)^2} \|f\|_{A^2}.$$

Proof: Write

$$f(w) = \sum_{k=m}^{\infty} a_k w^k$$

so that

$$\|f\|_{A^2}^2 = \sum_{k=m}^{\infty} \frac{|a_k|^2}{k+1}.$$

Using the Cauchy–Schwarz inequality we see that

$$|f'(w)| \leq |w|^{m-1} \Big(\sum_{k=0}^{\infty} (k+m)^2 (k+m+1) |w|^{2k} \Big)^{1/2} \|f\|_{A^2}.$$

Since $k+m+1 \leq 2(k+m)$, we get that

$$|f'(w)| \leq \sqrt{2} |w|^{m-1} \Big(\sum_{k=0}^{\infty} (k+m)^3 |w|^{2k} \Big)^{1/2} \|f\|_{A^2}. \tag{1.6}$$

Differentiating $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$ three times we see that

$$\frac{6}{(1-|w|^2)^4} = \sum_{k=0}^{\infty} (k+3)(k+2)(k+1)|w|^{2k}.$$

Since $k+m < km+3m = m(k+3)$, $k+m < km+2m = m(k+2)$, and $k+m < km+m = m(k+1)$, we see that

$$\sum_{k=0}^{\infty} (k+m)^3 |w|^{2k} \leq m^3 \sum_{k=0}^{\infty} (k+3)(k+2)(k+1)|w|^{2k} = 6m^3 (1-|w|^2)^{-4}.$$

From (1.6) we now get

$$|f'(w)| \leq |w|^{m-1} \frac{2\sqrt{3}m^{3/2}}{(1-|w|^2)^2} ||f||_{A^2},$$

as desired. ∎

2.4 Complex Interpolation. In addition to the material on analytic function spaces described above, we will also be using ideas from Alberto Calderón's theory of complex interpolation, as given in [7]. In this subsection we give the information that we need for our proofs of Theorems 9 and 10.

A pair of Banach spaces, $(X_0, ||\cdot||_{X_0})$ and $(X_1, ||\cdot||_{X_1})$, is called a "compatible pair (in the sense of Calderón)" if each is continuously embedded in some complex Hausdorff topological vector space, and if $X_0 \cap X_1$ is dense in each of X_0 and X_1. If X_0 and X_1 form a compatible pair of Banach spaces, one can construct other Banach spaces, indexed by the unit interval, according to the "method of complex interpolation". As is standard, we let $[X_0, X_1]_t$, $0 \leq t \leq 1$, denote this scale of Banach spaces. Now let T: $X_0 \cap X_1 \to X_0 \cap X_1$ be a linear map that is continuous with respect to both of the norms $||\cdot||_{X_0}$ and $||\cdot||_{X_1}$. Then it can be shown that T can be extended uniquely to a bounded linear operator on $[X_0, X_1]_t$, for each $0 \leq t \leq 1$. We will use the following result, relating the spectra of T on the intermediate spaces with the spectra of T on the endpoint spaces.

Theorem (part of [18] Theorem 2): *With notation as in the preceding paragraph,*

$$\sigma_{[X_0,X_1]_t}(T) \subseteq \sigma_{X_0}(T) \cup \sigma_{X_1}(T) \cup \sigma_{X_0\cap X_1}(T)$$

for each $0 \leq t \leq 1$. The last set in this union is the spectrum of T as viewed as an operator on the space $X_0 \cap X_1$, which is a Banach space when normed by

$$||f||_\cap \equiv \max\{||f||_{X_0}, ||f||_{X_1}\}.$$

If it is the case that $X_1 \subseteq X_0$ and there is a positive number K so that $||f||_{X_0} \leq K||f||_{X_1}$ for all $f \in X_1$, then any T invertible on X_1 must also be invertible on $X_0 \cap X_1$ (meaning that the inverse for T on X_1 also must be continuous with respect to the norm on $X_0 \cap X_1$). Hence, in this case,

$$\sigma_{X_0\cap X_1}(T) \subseteq \sigma_{X_1}(T)$$

and so

$$\sigma_{[X_0,X_1]_t}(T) \subseteq \sigma_{X_0}(T) \cup \sigma_{X_1}(T).$$

In the context of our work in this paper, we are motivated by the fact that for any fixed value of $p_0 \geq 1$, A^{p_0} and $\mathcal{B}$ form a compatible pair in the sense of Calderón, and for each $t \in [0,1]$,

$$[A^{p_0}, \mathcal{B}]_t = A^p \quad \text{(with equivalent norms)} \tag{1.7}$$

for p defined by $p_0 = p(1-t)$. In particular, $[A^1, \mathcal{B}]_t = A^p$ for $1 = p(1-t)$. Similarly, for each $t \in [0,1]$,

$$[H^1, BMOA]_t = H^p \quad \text{(with equivalent norms)} \tag{1.8}$$

for p defined by $1 = p(1-t)$. A proof of equation (1.7) is given in [27]. Equation (1.8) follows from Theorem 2 of [23], since it is known that $[H^1, H^p]_\phi = H^2$ for $\frac{1}{2} = (1-\phi)/1 + \phi/p$ (see, for example, [10]) and that $[H^2, BMOA]_\theta = H^p$ for $1/p = (1-\theta)/2$ (see, for example, page 191 of [25]). We thank Nigel Kalton for pointing us in the direction of Wolff's paper [23].

We point the reader in the direction of [3] or [25] if he/she feels the urge to read more about the general theory of complex interpolation.

3. Spectra of composition operators on Bergman and Bloch spaces

We have three main theorems; these will appear as Theorems 5, 8, and 9 of this section.

The first result relates the essential spectral radius of composition operators on various Bergman spaces. Recall from the introduction that for Hardy spaces this was done first by Bourdon and Shapiro in [6].

THEOREM 5: *Suppose that φ is an analytic map from the unit disk to itself. Then, for each $1 < p < \infty$,*

$$\left(r_{e,A^p}(C_\varphi)\right)^p = \left(r_{e,A^2}(C_\varphi)\right)^2.$$

The proof of Theorem 5 will follow readily from the next two propositions.

PROPOSITION 6: *Suppose that φ is an analytic map from the unit disk to itself. Then, for each $1 < p < \infty$,*

$$\left(r_{e,A^p}(C_\varphi)\right)^p \geq \left(r_{e,A^2}(C_\varphi)\right)^2.$$

Proof: Making use of the generalized Nevanlinna counting function, our proof follows the outline of the Hardy space case as in [6]. Let

$$f_a(z) = \left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{2/p}.$$

Since these are unit vectors in A^p which tend to 0 uniformly on compact subsets of U as $|a| \to 1^-$, they converge weakly to 0 in $A^p, p > 1$, and for arbitrary compact operator K on A^p, $\|Kf_a\|_{A^p} \to 0$ as $|a| \to 1^-$. Thus

$$\|C_\varphi\|^p_{e,A^p} \geq \limsup_{|a|\to 1^-} \|C_\varphi f_a\|^p_{A^p}.$$

The work of the proof will be to show that there is a constant $c(p)$, depending on p, such that

$$\limsup_{|a|\to 1^-} \|C_\varphi f_a\|^p_{A^p} \geq c(p)\|C_\varphi\|^2_{e,A^2}.$$

Upon replacing φ by its nth iterate φ_n, recalling $C_{\varphi_n} = C^n_\varphi$, and invoking the essential spectral radius formula we obtain the desired result from this estimate.

From formula (1.2) we get

$$\|C_\varphi f_a\|^p_{A^p} \approx |f_a(\varphi(0))|^p + \frac{16|a|^2}{p^2}(1-|a|^2)^2 \int_U \frac{1}{|1-\bar{a}z|^6} N_{\varphi,2}(z)dA(z).$$

Let, for the moment, I denote the second term on the right hand side. Suppose we can show that

$$I \geq k(p)\|C_\varphi\|^2_{e,A^2}$$

for some constant $k(p)$ depending on p. Then, if M is the smaller of the two constants in the definition of "$\approx$",

$$\|C_\varphi f_a\|^p_{A^p} \geq M\Big(|f_a(\varphi(0))|^p + k(p)\|C_\varphi\|^2_{e,A^2}\Big)$$

and this, in turn, is

$$\geq c(p)\|C_\varphi\|^2_{e,A^2},$$

for the constant $c(p) = Mk(p)$ depending on p. This observation gives us the freedom to forget about the term $|f_a(\varphi(0))|^p$ and focus on the term I.

If we set

$$v_a(z) = \frac{a-z}{1-\bar{a}z},$$

then

$$|v_a'(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$$

and so

$$I = \frac{16|a|^2}{p^2} \int_U \frac{1}{|1-\bar{a}z|^2} N_{\varphi,2}(z)|v_a'(z)|^2 dA(z).$$

Substituting $z = v_a(w)$, so that $w = v_a(z)$ and $dA(w) = |v_a'(z)|^2 dA(z)$, yields

$$\begin{aligned} I &= \frac{16|a|^2}{p^2} \int_U \frac{1}{|1-\bar{a}(\frac{a-w}{1-\bar{a}w})|^2} N_{\varphi,2}(v_a(w))dA(w) \\ &= \frac{16|a|^2}{p^2} \int_U \frac{|1-\bar{a}w|^2}{(1-|a|^2)^2} N_{\varphi,2}(v_a(w))dA(w). \end{aligned}$$

Then, for any $0 < r < 1$, we get $|1-\bar{a}w| \geq 1-r$ for $w \in rU$ and hence

$$\begin{aligned} I &\geq \frac{16|a|^2}{p^2} \int_{rU} \frac{|1-\bar{a}w|^2}{(1-|a|^2)^2} N_{\varphi,2}(v_a(w))dA(w) \\ &\geq \frac{16|a|^2(1-r)^2}{p^2(1-|a|^2)^2} \int_{rU} N_{\varphi,2}(v_a(w))dA(w). \end{aligned}$$

We now fix $0 < r < 1$ and consider $|a|$ close to 1. We need $|a|$ close enough to 1 to ensure that $\varphi(0) \notin v_a(rU)$. Then $N_{\varphi,2}(v_a(w))$ satisfies a sub-mean value property on rU ([19] Corollary 6.7), so that

$$\int_{rU} N_{\varphi,2}(v_a(w))dA(w) \geq r^2 N_{\varphi,2}(v_a(0)) = r^2 N_{\varphi,2}(a).$$

Therefore,

$$I \geq \frac{16|a|^2(1-r)^2 r^2}{p^2(1-|a|^2)^2} N_{\varphi,2}(a).$$

Observe that $1-|a|$ and $\log(1/|a|)$ are comparable as $|a| \to 1^-$, and therefore can be used interchangeably. Taking $\limsup_{|a|\to 1^-}$ on both sides and using formula (1.5) thus gives

$$\limsup_{|a|\to 1^-} \|C_\varphi f_a\|_{e,A^p}^p \geq c(p)\|C_\varphi\|_{e,A^2}^2,$$

as desired. ∎

While the proof of Proposition 6 is similar to the Hardy space proof, the proof of Proposition 7 differs in spirit from the proof for the Hardy space case. For instance, we do not have Blaschke factors available.

PROPOSITION 7: *Suppose that φ is an analytic map from the unit disk to itself that fixes a point in the interior of the disk. Then, for each $1 \leq p < \infty$,*

$$\left(r_{e,A^p}(C_\varphi)\right)^p \leq \left(r_{e,A^2}(C_\varphi)\right)^2.$$

Proof: If $a \in U$ and $\varphi(a) = a$ then C_φ is similar to a composition operator whose symbol fixes the origin. Because similar operators have the same essential spectral radius, there is no loss of generality in assuming that $\varphi(0) = 0$. Recall the material of the preliminary section 2.2. Observe that $\lim_{m\to\infty} \|C_\varphi\|_{A_m}$ exists, since the norms are decreasing. We claim that

$$\lim_{m\to\infty} \|C_\varphi\|_{A_m} \leq c\|C_\varphi\|_{e,A^2}^{2/p}. \tag{1.9}$$

Here, and throughout the rest of the argument, c will denote a constant whose value may change from time to time but always only depends on p. To see that the result follows from the claim, note that if $|\lambda| > \|C_\varphi\|_{A_m}$ for some m, then $C_\varphi - \lambda I$ is invertible on A_m. By Lemma 3.5 of [6], $C_\varphi - \lambda I$ is Fredholm on A^p. Therefore,

$$r_{e,A^p}(C_\varphi) \leq \inf_{m\in\mathbb{N}} \|C_\varphi\|_{A_m} = \lim_{m\to\infty} \|C_\varphi\|_{A_m}.$$

Hence, assuming the claim, we have

$$r_{e,A^p}(C_\varphi) \leq c\|C_\varphi\|_{e,A^2}^{2/p},$$

whenever $\varphi(0) = 0$. Proceeding as in the proof of Theorem 3.8 of [6], replacing φ by its nth iterate φ_n and taking nth roots, we obtain

$$r_{e,A^p}(C_{\varphi_n})^{1/n} \leq c^{1/n}\left[\left\|C_{\varphi_n}\right\|_{e,A^2}^{2/p}\right]^{1/n}. \tag{1.10}$$

The left-hand side of this is equal to $r_{e,A^p}(C_\varphi)$, since $r_e(T^n) = (r_e(T))^n$ for any bounded linear operator T and since $C_{\varphi_n} = C_\varphi^n$. Letting $n \to \infty$, the right-hand side of (1.10) has limit $(r_{e,A^2}(C_\varphi))^{2/p}$, by the spectral radius formula. We conclude that

$$r_{e,A^p}(C_\varphi) \leq (r_{e,A^2}(C_\varphi))^{2/p}.$$

Thus we need only establish the claim (1.9).

Since by equation (1.5) we know

$$\|C_\varphi\|_{e,A^2}^2 = \limsup_{|a|\to 1^-} \frac{N_{\varphi,2}(a)}{(\log(1/|a|))^2},$$

it suffices to show that

$$\lim_{m\to\infty} \|C_\varphi\|_{A_m}^p \leq c \limsup_{|a|\to 1^-} \frac{N_{\varphi,2}(a)}{(\log(1/|a|))^2}.$$

Now

$$\|C_\varphi\|_{A_m}^p = \sup\{\|f\circ\varphi\|_{A^p}^p \colon f \in A_m,\ \|f\|_{A^p} = 1\}$$

which, by equation (1.2) and the hypothesis $\varphi(0) = 0$, is bounded above by a constant multiple of

$$\sup\left\{\int_U |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,2}(z)dA(z)\colon f \in A_m,\ \|f\|_{A^p} = 1\right\}$$

for any $m \in \mathbb{N}$. For a fixed $0 < r < 1$ we first show that

(1.11)

$$\lim_{m\to\infty}\left[\sup\left\{\int_{rU} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,2}(z)dA(z)\colon f \in A_m,\ \|f\|_{A^p} = 1\right\}\right] = 0$$

by considering the cases $p > 2, p = 2, 1 < p < 2$ and $p = 1$ in turn. In doing this, we will use Propositions 1 and 2 to assert that, on rU,

$$|f(z)| \le c\frac{2m+1}{(1-r)^{2/p}} \cdot r^m \quad \text{and} \quad |f'(z)| \le c\frac{1}{(1-r)^{1+2/p}}$$

when f is a unit vector in A_m.

When $p > 2$ this leads to the estimate, for functions in the unit ball of A_m,

$$\begin{aligned}\int_{rU} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,2}(z)dA(z) &\le c\frac{(2m+1)^{p-2}}{(1-r)^4} \cdot r^{m(p-2)} \int_{rU} N_{\varphi,2}(z)dA(z)\\ &\le c\frac{(2m+1)^{p-2}}{(1-r)^4} \cdot r^{m(p-2)},\end{aligned}$$

since $N_{\varphi,2}(z) \le (\log 1/|z|)^2$ on $U \setminus \{0\}$ by Proposition 6.3 of [19], since $\varphi(0) = 0$. Thus (1.11) holds for $p > 2$.

The case $p = 2$ is handled in a similar, but easier, manner using Proposition 4.

When $1 < p < 2$, we begin with

$$\int_{rU} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,2}(z)dA(z) \le \int_{rU} |f(z)|^{p-2}|f'(z)|^2(\log 1/|z|)^2 dA(z),$$

which again follows from $N_{\varphi,2}(z) \le (\log(1/|z|))^2$ on $U \setminus \{0\}$. Next set $s = \sqrt{r}$ so $\log(1/|z|) \le 2\log(s/|z|)$ on rU. Thus

(1.12)

$$\int_{rU} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,2}(z)dA(z) \le 4\int_{rU} |f(z)|^{p-2}|f'(z)|^2\left(\log\frac{s}{|z|}\right)^2 dA(z).$$

Find $1 < p_1 < p < 2$ and write the integrand on the right-hand side of (1.12) as

$$\left(|f(z)|^{p_1-2}|f'(z)|^2 \log\frac{s}{|z|}\right)\left(|f(z)|^{p-p_1}\log\frac{s}{|z|}\right).$$

In the second factor we make the estimate

$$|f(z)| \le c\frac{2m+1}{(1-|z|)^{2/p}} \cdot |z|^m$$

for f in the unit ball of A_m and then observe that

$$|z|^{m(p-p_1)} \log \frac{s}{|z|} \leq |z|^{m(p-p_1)} \log \frac{1}{|z|}.$$

This is bounded by 1 on rU for m sufficiently large. Thus, for such sufficiently large m, the right-hand side of (1.12) is bounded above by

$$\begin{aligned} 4\Big(c\frac{2m+1}{(1-r)^{2/p}}\Big)^{p-p_1} &\int_{rU} |f(z)|^{p_1-2}|f'(z)|^2 \log \frac{s}{|z|} dA(z) \\ &\leq c\Big(\frac{2m+1}{(1-r)^{2/p}}\Big)^{p-p_1} \int_{sU} |f(z)|^{p_1-2}|f'(z)|^2 \log \frac{s}{|z|} dA(z) \\ &= c\Big(\frac{2m+1}{(1-r)^{2/p}}\Big)^{p-p_1} \frac{2}{p_1^2} \int_0^{2\pi} |f(se^{i\theta})|^{p_1} \frac{d\theta}{2\pi}. \end{aligned}$$

In this computation we have used $s > r$, the non-negativity of the integrand, and equation (1.3). Finally, since $f \in A_m$ is a unit vector we use Proposition 2 to see that

$$\int_0^{2\pi} |f(se^{i\theta})|^{p_1} \frac{d\theta}{2\pi} \leq c\Big(\frac{2m+1}{(1-s)^{2/p}} \cdot s^m\Big)^{p_1}$$

and hence to conclude that

$$\int_{rU} |f(z)|^{p-2}|f'(z)|^2 N_{\varphi,2}(z) dA(z) \leq c\frac{1}{(1-\sqrt{r})^2}(2m+1)^p \cdot r^{mp_1/2}.$$

Again, r is fixed so this tends to 0 as $m \to \infty$, verifying (1.11) for $1 < p < 2$.

Finally, when $p = 1$ we use similar computations. We have Equation (1.12) with $p - 2 = -1$, and the integrand on the right-hand side of (1.12) is similarly written as

$$\Big(|f(z)|^{-1-\epsilon}|f'(z)|^2 \log \frac{s}{|z|}\Big)\Big(|f(z)|^{\epsilon} \log \frac{s}{|z|}\Big)$$

for small positive ϵ. When f is in the unit ball of A_m for sufficiently large m we have

$$|f(z)|^{\epsilon} \log \frac{s}{|z|} \leq c\Big(\frac{2m+1}{(1-r)^{2/p}}\Big)^{\epsilon}$$

on rU. This leads to the estimate

$$\int_{rU} |f(z)|^{-1}|f'(z)|^2 N_{\varphi,2}(z) dA(z) \leq c\Big(\frac{2m+1}{(1-r)^{2/p}}\Big)^{\epsilon} \frac{2}{(1-\epsilon)^2} \int_0^{2\pi} |f(se^{i\theta})|^{1-\epsilon} \frac{d\theta}{2\pi}$$

and the argument is completed again by using the estimate

$$|f(se^{i\theta})| \leq c\frac{2m+1}{(1-s)^{2/p}} s^m$$

for f a unit vector in A_m. This finishes the verification of (1.11).

To complete the proof we turn to

$$\int_{U\setminus rU}|f(z)|^{p-2}|f'(z)|^2N_{\varphi,2}(z)dA\leq M_r\int_{U\setminus rU}|f(z)|^{p-2}|f'(z)|^2(\log|z|)^2dA,$$

where

$$M_r\equiv\sup\left\{\frac{N_{\varphi,2}(z)}{(\log|z|)^2}:r<|z|<1\right\}.$$

Replacing the integral on the right-hand side of the inequality by the integral over U and using

$$\int_U|f(z)|^{p-2}|f'(z)|^2(\log|z|)^2dA(z)\leq c\|f\|^p_{A^p}\leq c$$

for f in the unit ball of A_m, we see that

$$\int_{U\setminus rU}|f(z)|^{p-2}|f'(z)|^2N_{\varphi,2}(z)dA(z)\leq M_rc.$$

Letting $r\to 1^-$ and recalling (1.11) we see that

$$\lim_{m\to\infty}\|C_\varphi\|^p_{A_m}\leq c\limsup_{|z|\to 1^-}\frac{N_{\varphi,2}(z)}{(\log|z|)^2}$$

as desired. ∎

Proof of Theorem 5: When φ has a fixed point in U, the result follows immediately from Propositions 6 and 7. If φ has no fixed point in U, then it has a Denjoy–Wolff point on the unit circle ∂U (see, for example, section 2.3 of [9]). In this case, only the inequality

$$\left(r_{e,A^p}(C_\varphi)\right)^p\leq\left(r_{e,A^2}(C_\varphi)\right)^2$$

needs verification since Proposition 6 gives the reverse inequality. A proof can be constructed using the following steps. Simple modifications to the proof of Theorem 3.9 in [9] show that if φ has Denjoy–Wolff point $w\in\partial U$ then for $p\geq 1$, $r_{A^p}(C_\varphi)=(\varphi'(w))^{-2/p}$ so that $(r_{A^p}(C_\varphi))^p=(r_{A^2}(C_\varphi))^2$. Since

$$r_{e,A^p}(C_\varphi)\leq r_{A^p}(C_\varphi)=(r_{A^2}(C_\varphi))^{2/p},$$

we can finish an argument similar to that in Lemma 5.2 of [6] showing that $r_{e,A^2}(C_\varphi)=r_{A^2}(C_\varphi)$ when φ has Denjoy–Wolff point on ∂U. ∎

The next result identifies the spectrum of C_φ acting on A^p, $1 \le p < \infty$, when φ is univalent, is not an automorphism, and fixes a point of U. The same result for a variety of Hilbert spaces of analytic functions including H^2 and A^2 was obtained in [8]. The argument we use is very similar to that in [8] and makes use of the behavior of the iteration sequences of $\varphi\colon U \to U$ when $\varphi(0) = 0$. Recall that $\{z_k\}$ is an iteration sequence for φ if $\varphi(z_k) = z_{k+1}$ for all k. We will need the following two lemmas from [8] which describe the behavior of the iteration sequences when $\varphi(0) = 0$.

LEMMA ([8] Lemma 14, [9] Lemma 7.35): *Suppose $\varphi\colon U \to U$ is analytic, not an automorphism, and fixes the origin. There exists $b < 1$ so that for any iteration sequence $\{z_k\}$ we have*

$$\frac{|z_{k+1}|}{|z_k|} \le b \tag{1.13}$$

whenever $|z_k| \le \frac{1}{2}$.

LEMMA ([8] Lemma 13, [9] Lemma 7.34): *Suppose $\varphi\colon U \to U$ is analytic, not an automorphism, and fixes the origin. Let $0 < r < 1$. There exists $1 \le M < \infty$ so that if $\{z_k\}_{-K}^{\infty}$ is an iteration sequence with $|z_n| \ge r$ for some non-negative integer n and $\{w_k\}_{-K}^{n}$ are arbitrary complex numbers, then there exists $f \in H^\infty$ with*

$$f(z_k) = w_k, \quad -K \le k \le n,$$

and

$$\|f\|_{H^\infty} \le M \sup\{|w_k| : -K \le k \le n\}.$$

We next standardize our indexing notation for the iteration sequences when $\varphi(0) = 0$. Henceforth, iteration sequences will be denoted $\{z_k\}_{-K}^{\infty}$ where K is a positive integer and $|z_0| \ge \frac{1}{2}$. Determine the non-negative integer n by

$$n = \max\{k : |z_k| \ge \frac{1}{4}\}. \tag{1.14}$$

Note that $|z_k| < \frac{1}{4}$ for $k > n$ and there exists a $b < 1$, by the first lemma, so that $|z_{k+1}|/|z_k| \le b$ for all $k \ge n$. We may assume that $\frac{1}{2} \le b < 1$. By repeated application of this we get $|z_k| \le |z_n| b^{k-n}$ for all $k \ge n$.

We can now give the main result on spectra of composition operators on A^p.

THEOREM 8: *Suppose that φ is univalent, not an automorphism, with fixed point $a \in U$. Then, for $1 \le p < \infty$,*

$$\sigma_{A^p}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \le r_{e,A^p}(C_\varphi)\} \cup \{(\varphi'(a))^n\}_{n=0}^{\infty}.$$

Proof: Without loss of generality, we may assume that $a = 0$, since if $\varphi(a) = a$ then C_φ is similar to a composition operator whose symbol fixes 0, and similar operators have the same spectrum. Whenever $\varphi(0) = 0$ and C_φ acts on any space of analytic functions which contains the polynomials, then $\{(\varphi'(0))^n\}_{n=0}^\infty$ is contained in the spectrum ([12] Lemma 2). If $\lambda \in \sigma_{A^p}(C_\varphi)$ and $|\lambda| > r_{e,A^p}(C_\varphi)$ then λ is an eigenvalue of C_φ (see, for example, Proposition 2.2 of [6]). By Koenig's Theorem (see, for example, Theorem 2.63 of [9]) the eigenvalues of C_φ must be of form $(\varphi'(0))^n$ for some non-negative integer n. Thus we need only show that

$$\{\lambda \in \mathbb{C} : |\lambda| \le r_{e,A^p}(C_\varphi)\} \subseteq \sigma_{A^p}(C_\varphi).$$

If $r_{e,A^p}(C_\varphi) = 0$ there is nothing to show since $0 \in \sigma_{A^p}(C_\varphi)$ whenever φ is not an automorphism. Thus for the rest of the argument we assume $\rho \equiv r_{e,A^p}(C_\varphi) > 0$ and choose λ, $0 < |\lambda| < \rho$. If we can show $(C_\varphi - \lambda I)^*$ is not invertible we will be done, since the spectrum is closed. Indeed, we will show that $(C_m - \lambda I)^*$ is not invertible, where C_m is the restriction of C_φ to the invariant subspace A_m. By the comments of subsection 2.2, once we know that $\lambda \in \sigma(C_m) \equiv \sigma_{A_m}(C_\varphi)$ we can conclude that $\lambda \in \sigma_{A^p}(C_\varphi)$.

We next fix a suitable value of m. Recall the constants b, $\frac{1}{2} \le b < 1$, of the first lemma (so that (1.13) is satisfied for any iteration sequence as just described), and M, $1 \le M < \infty$, of the second lemma using $r = \frac{1}{4}$ and our chosen $\lambda \ne 0$. Fix m sufficiently large so that

$$\frac{b^m}{|\lambda|} < \frac{1}{81M}. \tag{1.15}$$

For any iteration sequence, $\{z_k\}_{k=-K}^\infty$, we can define a linear functional L_λ on A_m by

$$L_\lambda(f) = \sum_{k=-K}^\infty \lambda^{-k} f(z_k).$$

This is bounded since, for n as defined by (1.13), Proposition 2 gives for arbitrary $f \in A_m$,

$$\begin{aligned}\Big|\sum_{k=n+1}^\infty \lambda^{-k} f(z_k)\Big| &\le c(p) \sum_{k=n+1}^\infty |\lambda|^{-k} \frac{2m+1}{(1-|z_k|^2)^{2/p}} \cdot |z_k|^m \cdot \|f\|_{A^p} \\ &\le c(p)\Big(\frac{16}{15}\Big)^{2/p}(2m+1)\|f\|_{A^p} \sum_{k=n+1}^\infty \frac{|z_k|^m}{|\lambda|^k}.\end{aligned}$$

This is finite, since $|z_k| \le |z_n| b^{k-n}$ for $k \ge n$ and $b^m/|\lambda| < 1$.

A straightforward calculation shows that

$$(C_m^* - \lambda)L_\lambda = -\lambda^{K+1} ev_{z_{-K}}.$$

Our goal is to show that $C_m^* - \lambda$ is not bounded below by considering its action on $L_\lambda/\|L_\lambda\|_{A_m}$, where L_λ is defined from a judiciously chosen iteration sequence. To carry this out we need a lower bound estimate on $\|L_\lambda\|_{A_m}$. Given any iteration sequence $\{z_k\}_{k=-K}^{\infty}$, with $n \geq 0$ defined by (1.14), we know by the second lemma that there exists $f \in H^\infty$, $\|f\|_{H^\infty} \leq M$, satisfying

(i) $|f(z_k)| = 1$ for $k = 0$ and $k = n$,
(ii) $z_k^m f(z_k)/(\lambda^k(1 - \overline{z_0}z_k)^{4/p}) > 0$ for $k = 0$ and $k = n$,
(iii) $f(z_k) = 0$ for $-K \leq k < n$, $k \neq 0$.

For such f we calculate

$$L_\lambda\Big(\frac{z^m f(1 - |z_0|^2)^{2/p}}{(1 - \overline{z_0}z)^{4/p}}\Big) = \sum_{k=-K}^{\infty} \lambda^{-k} \frac{z_k^m f(z_k)(1 - |z_0|^2)^{2/p}}{(1 - \overline{z_0}z_k)^{4/p}}.$$

Note that the terms of this sum corresponding to $-K \leq k \leq n$ are zero unless $k = 0$ or $k = n$. The $k = 0$ term is $|z_0|^m/(1 - |z_0|^2)^{2/p}$ and the $k = n$ term is

$$\frac{|z_n|^m}{|\lambda|^n|1 - \overline{z_0}z_n|^{4/p}} \geq \frac{|z_n|^m}{16|\lambda|^n}.$$

We also have the estimate

$$\begin{aligned}\Big|\sum_{k=n+1}^{\infty} \frac{z_k^m f(z_k)(1 - |z_0|^2)^{2/p}}{\lambda^k(1 - \overline{z_0}z_k)^{4/p}}\Big| &\leq \Big(\frac{4}{3}\Big)^4 \cdot \frac{M|z_n|^m(1 - |z_0|^2)^{2/p}}{|\lambda|^n} \sum_{k=n+1}^{\infty} \Big(\frac{b^m}{|\lambda|}\Big)^{k-n} \\ &\leq \frac{|z_n|^m}{16|\lambda|^n},\end{aligned}$$

where we have used $\|f\|_{H^\infty} \leq M$, $|z_k| \leq \frac{1}{4}$, and $|z_k| \leq |z_n| b^{k-n}$ for $k \geq n + 1$ in the first inequality and

$$\sum_{k=n+1}^{\infty} \Big(\frac{b^m}{|\lambda|}\Big)^{k-n} < \frac{\frac{1}{81M}}{1 - \frac{1}{81M}} \leq \frac{1}{80M}$$

in the second. By these estimates and the reverse triangle inequality we see that

$$\Big|L_\lambda\Big(\frac{z^m f(1 - |z_0|^2)^{2/p}}{(1 - \overline{z_0}z)^{4/p}}\Big)\Big| \geq \frac{|z_0|^m}{(1 - |z_0|^2)^{2/p}}.$$

But for $1 \leq p < \infty$

$$\Big\|\frac{z^m f(1 - |z_0|^2)^{2/p}}{(1 - \overline{z_0}z)^{4/p}}\Big\|_{A^p} \leq M$$

since

$$\frac{(1-|z_0|^2)^{2/p}}{(1-\overline{z_0}z)^{4/p}}$$

is a unit vector in A^p. This gives the desired lower bound on $\|L_\lambda\|_{A_m}$; indeed, for

$$g(z) \equiv \frac{(1-|z_0|^2)^{2/p}}{(1-\overline{z_0}z)^{4/p}}$$

we have

$$\begin{aligned}\|L_\lambda\|_{A_m} &\geq \frac{\left|L_\lambda\left(z^m fg\right)\right|}{\left\|z^m fg\right\|_{A^p}} \\ &\geq \frac{1}{M}\cdot |z_0|^m(1-|z_0|^2)^{-2/p} \\ &\geq \frac{1}{M\cdot c(p,m)}\cdot \|ev_{z_0}\|_{A_m}\end{aligned}$$

where the last inequality, with $c(p,m)$ denoting a constant depending only on p and m, follows from Proposition 2.

We are now ready to make a judicious choice of iteration sequence. Recall that

$$r_{e,A^p}(C_\varphi) = \lim_{n\to\infty} \|C_{\varphi_n}\|_{e,A^p}^{1/n} = \inf_{n\in\mathbb{N}} \|C_{\varphi_n}\|_{e,A^p}^{1/n}.$$

Since φ is univalent we know that

$$\|C_{\varphi_n}\|_{e,A^2}^2 = \limsup_{|w|\to 1^-}\left(\frac{1-|\varphi_n^{-1}(w)|}{1-|w|}\right)^2. \tag{1.16}$$

Whenever this is non-zero it may be equivalently calculated as

$$\|C_{\varphi_n}\|_{e,A^2}^2 = \limsup_{|w|\to 1^-}\left(\frac{1-|w|}{1-|\varphi_n(w)|}\right)^2.$$

Given $0<|\lambda|<\rho\equiv r_{e,A^p}(C_\varphi)$, find ρ' such that $|\lambda|<\rho'<\rho$. Since by Proposition 7

$$\rho = \lim_{n\to\infty} \|C_{\varphi_n}\|_{e,A^p}^{1/n} \leq \lim_{n\to\infty}\left(\|C_{\varphi_n}\|_{e,A^2}^{2/p}\right)^{1/n}$$

there exists N_0 such that for all $n\geq N_0$

$$\left(\|C_{\varphi_n}\|_{e,A^2}^{2/p}\right)^{1/n} > \rho'$$

or

$$\|C_{\varphi_n}\|_{e,A^2}^{2/p} > (\rho')^n.$$

Thus, for any $K \geq N_0$ we can find, by (1.16), a w sufficiently close to ∂U so that

(i) $\left(\frac{1-|w|}{1-|\varphi_K(w)|}\right)^{2/p} \geq (\rho')^K$,
(ii) $|\varphi_K(w)| \geq \frac{1}{2}$, and
(iii) $\frac{\|ev_{\varphi_K(w)}\|_{A_m}}{\|ev_w\|_{A_m}} \geq \frac{1}{2^m}\frac{\|ev_{\varphi_K(w)}\|_{A^p}}{\|ev_w\|_{A^p}}$.

The inequality of (iii) follows from (ii) and Proposition 3.

Facts (i) and (iii) together with Proposition 1 give

$$\frac{\|ev_{\varphi_K(w)}\|_{A_m}}{\|ev_w\|_{A_m}} \geq \frac{1}{2^m}(\rho')^K.$$

With this choice of w we form an iteration sequence $\{z_k\}_{-K}^{\infty}$ by setting $z_{-K} = w$ so that $z_0 = \varphi_K(w)$ and $|z_0| \geq \frac{1}{2}$. At this point the positive integer K is still arbitrary, except for the requirement that $K \geq N_0$. Our estimates say

$$\begin{aligned}\frac{\|(C_m^* - \lambda)L_\lambda\|_{A_m}}{\|L_\lambda\|_{A_m}} &\leq \frac{|\lambda|^{K+1}\|ev_{z_{-K}}\|_{A_m}}{\|ev_{z_0}\|_{A_m}} \cdot M \cdot c(m,p)\\ &\leq |\lambda|^{K+1} \cdot M \cdot c(m,p)\left(\frac{1}{\rho'}\right)^K\\ &\leq |\lambda| \cdot M \cdot c(m,p)\left(\frac{|\lambda|}{\rho'}\right)^K.\end{aligned}$$

The various constants depend on (at most) p and m which are fixed values. We may thus choose $K \geq N_0$ large so that this product is as small as desired. This shows that $C_m^* - \lambda$ is not bounded below. ∎

If φ satisfies the hypotheses of the preceding theorem, then $r_{e,A^p}(C_\varphi) < 1$. We will have more to say about this fact in the last section, where we will prove it.

THEOREM 9: *Suppose that φ is univalent, is not an automorphism, and fixes a point in U. If $r_{e,A^p}(C_\varphi) \neq 0$ for any, and hence every, value of $1 < p < \infty$, then the spectrum of C_φ on the Bloch space, $\sigma_{\mathcal{B}}(C_\varphi)$, contains the annulus*

$$\{\lambda \in \mathbb{C} : r_{e,A^1}(C_\varphi) \leq |\lambda| \leq 1\}.$$

The inner radius, $r_{e,A^1}(C_\varphi)$, is at most $(r_{e,A^2}(C_\varphi))^2$.

Proof: We rely on the material described in subsection 2.4.

Recall that $[A^{p_0}, \mathcal{B}]_t$, $0 \leq t \leq 1$, denotes the scale of Banach spaces constructed via Calderón's method of complex interpolation. An application of Theorem 2 of [18] gives

$$\sigma_{[A^{p_0},\mathcal{B}]_t}(C_\varphi) \subseteq \sigma_{A^{p_0}}(C_\varphi) \cup \sigma_{\mathcal{B}}(C_\varphi) \cup \sigma_{A^{p_0}\cap\mathcal{B}}(C_\varphi)$$

for all $1 \leq p_0 < \infty$ and $0 \leq t \leq 1$. Equation (1.7) promises that $[A^{p_0}, \mathcal{B}]_t$ will be the same set of functions that are in A^p (where t and p have the appropriate relation), but the norm on $[A^{p_0}, \mathcal{B}]_t$ is only promised to be equivalent to the standard norm defined on A^p. However, changing from one equivalent complete norm to another does not give any change in the spectrum of an operator on the space. Thus, Theorem 2 of [18] gives

$$\sigma_{A^p}(C_\varphi) \subseteq \sigma_{A^{p_0}}(C_\varphi) \cup \sigma_{\mathcal{B}}(C_\varphi) \cup \sigma_{A^{p_0} \cap \mathcal{B}}(C_\varphi)$$

for all $1 \leq p_0 < p < \infty$.

Next, observe that $\mathcal{B} \subseteq A^p$ for all $p < \infty$ and that there is a positive number K (depending on p) such that $\|f\|_{A^p} \leq K\|f\|_{\mathcal{B}}$ for each $f \in \mathcal{B}$. This follows from the estimate

$$|f(z) - f(0)| \leq \frac{1}{2} \log \frac{1+|z|}{1-|z|} \|f\|_{\mathcal{B}}$$

which holds for all $f \in \mathcal{B}$ (see, for example, Theorem 5.1.6 in [25]). Therefore,

$$\sigma_{A^p}(C_\varphi) \subseteq \sigma_{A^{p_0}}(C_\varphi) \cup \sigma_{\mathcal{B}}(C_\varphi)$$

for all $1 \leq p_0 < p < \infty$.

Specializing to the choice $p_0 = 1$ and using Theorem 8 and the fact that the spectrum is closed, we have

$$\{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,A^p}(C_\varphi)\} \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,A^1}(C_\varphi)\} \cup \sigma_{\mathcal{B}}(C_\varphi)$$

for all $1 \leq p < \infty$. From our Theorem 5 we have that

$$r_{e,A^p}(C_\varphi) \uparrow 1, \quad \text{as } p \uparrow \infty.$$

Therefore,

$$\{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,A^1}(C_\varphi)\} \cup \sigma_{\mathcal{B}}(C_\varphi).$$

Since the spectrum is always a closed set and $r_{e,A^1}(C_\varphi) < 1$, we get the desired description of $\sigma_{\mathcal{B}}(C_\varphi)$. By Proposition 7 we know that

$$r_{e,A^1}(C_\varphi) \leq (r_{e,A^2}(C_\varphi))^2. \qquad \blacksquare$$

When the hypotheses of Theorem 9 hold, then the essential norm of C_φ on the Bloch space must equal 1. This fact holds under more general hypotheses as well (for example, if φ has finite angular derivative at some point of U); see

Theorem 2.1 of [14] or chapter 5 of [11]. When $r_{e,A^2}(C_\varphi) = 0$ the essential norm on the Bloch space can be strictly less than 1. Indeed, for any $0 < s < 1$ there exist univalent φ fixing 0 with $\|C_\varphi\|_{e,\mathcal{B}} = s$ (see the discussion of the "lens map" in Proposition 6.4 of [11] and the equality of essential norm of a composition operator on the Bloch and little Bloch spaces in Theorem 2.1 of [14]). While these examples satisfy $r_{e,A^2}(C_\varphi) = 0$ since C_φ is compact on A^2, it is possible for the essential spectral radius on A^2 to be 0 even though C_φ is not compact on A^2. For a specific example, notice that the univalent φ with C_φ not compact on H^2 yet $r_{e,H^2}(C_\varphi) = 0$ given in [6] also provides an example with the same properties on A^2.

4. Spectra of composition operators on Hardy spaces and BMOA

There is an analogue of Theorem 9 for the spectrum of C_φ acting on BMOA when φ is univalent, is not an automorphism, and fixes a point in U.

THEOREM 10: *Suppose that φ is univalent, is not an automorphism, and fixes a point in U. If $r_{e,H^p}(C_\varphi) \neq 0$ for any, and hence every, value of $1 < p < \infty$, then the spectrum of C_φ on BMOA, $\sigma_{BMOA}(C_\varphi)$, contains the annulus*

$$\{\lambda \in \mathbb{C} : r_{e,H^1}(C_\varphi) \leq |\lambda| \leq 1\}$$

where the inner radius $r_{e,H^1}(C_\varphi)$ is at most $(r_{e,H^2}(C_\varphi))^2$.

We briefly summarize the modifications needed to some of the results of sections 2.3 and 3 in order to prove Theorem 10.

The H^p version of Proposition 1 is well-known (see, for example, Corollary 2.14 in [9]). In part (a) the exponent $2/p$ is replaced by $1/p$. A similar use of Cauchy's formula gives the H^p version of (b): for $w \in U$,

$$|f'(w)| \leq c(p)(1 - |w|)^{-(p+1)/p}\|f\|_{H^p}.$$

There is an H^p version of Proposition 2 which is both stronger and easier to obtain. This is our next result.

PROPOSITION 11: *For $w \in U$ and $p \geq 1$,*

$$\|ev_w\|_{H_m} \approx |w|^m(1 - |w|^2)^{-1/p}$$

where $H_m = z^m H^p$, m any positive integer.

Proof: Since $f \in z^m H^p$ implies $f = z^m g$ for some $g \in H^p$ with $\|g\|_{H^p} = \|f\|_{H^p}$, it follows that

$$|f(w)| = |w|^m|g(w)| \leq c(p)|w|^m(1 - |w|)^{-1/p}\|f\|_{H^p}.$$

Using the H^p unit vectors $f(z) = z^m(1-|w|^2)^{1/p}(1-\overline{w}z)^{-2/p}$ we get the other inequality. ∎

The next result is the Hardy space analogue of Proposition 3.

PROPOSITION 12: *Let $1 \leq p < \infty$. There is a constant $c(p)$ so that for all $w \in U$ with $|w| \geq \frac{1}{2}$,*

$$\|ev_w\|_{H_m} \leq \|ev_w\|_{H^p} \leq c(p)2^m\|ev_w\|_{H_m}.$$

Proof: Only the right hand inequality needs proof. This follows from $\|ev_w\|_{H^p} \approx (1-|w|)^{-1/p}$ and

$$\|ev_w\|_{H_m} \approx |w|^m(1-|w|)^{-1/p} \geq c(p)2^{-m}(1-|w|)^{-1/p}$$

for $|w| \geq \frac{1}{2}$. ∎

The version of Proposition 4 for z^mH^2 appears as Lemma 5.2(b) of [20]: For every $z \in U$,

$$|f'(z)| \leq \sqrt{2}m|z|^{m-1}(1-|z|^2)^{-3/2}\|f\|_{H^2}.$$

We next prove an H^p version of Theorem 8, identifying the spectrum when φ, not an automorphism, is univalent with a fixed point in U.

THEOREM 13: *Suppose φ is univalent, not an automorphism, with fixed point a in U. Then for $1 \leq p < \infty$,*

$$\sigma_{H^p}(C_\varphi) = \{\lambda \in C : |\lambda| \leq r_{e,H^p}(C_\varphi)\} \cup \{(\varphi'(a))^n\}_{n=0}^{\infty}.$$

Proof: The outline of the argument is exactly the same as in the proof of Theorem 8, so we only indicate the necessary modifications. The linear functional L_λ acting on $H_m = z^mH^p$ (for fixed m chosen to satisfy inequality (1.14)) is defined by the same formula:

$$L_\lambda(f) = \sum_{k=-K}^{\infty} \lambda^{-k}f(z_k)$$

where $\{z_k\}_{-K}^{\infty}$ is an iteration sequence, indexed with the same conventions as described prior to the proof of Theorem 8. Proposition 11 gives the boundedness of L_λ on H_m.

We get a lower bound estimate on $\|L_\lambda\|_{H_m}$ by considering $L_\lambda(z^mfg)$ where g is the H^p unit vector

$$g(z) \equiv (1-|z_0|^2)^{1/p}(1-\overline{z_0}z)^{-2/p}$$

and f is an H^∞ function with $\|f\|_{H^\infty} \le M$ satisfying

(i) $|f(z_k)| = 1$ for $k = 0$ and $k = n$,
(ii) $(z_k^m f(z_k)/(\lambda^k(1-\overline{z_0}z_k)^{2/p})) > 0$ for $k = 0$ and $k = n$,
(iii) $f(z_k) = 0$ for $-K \le k < n$, $k \ne 0$.

Then

$$\begin{aligned} &L_\lambda(z^m fg) \\ &\quad= \sum_{k=-K}^{\infty} \lambda^{-k} \frac{z_k^m f(z_k)(1-|z_0|^2)^{1/p}}{(1-\overline{z_0}z_k)^{2/p}} \\ &\quad= \frac{|z_0|^m}{(1-|z_0|^2)^{1/p}} + \frac{|z_n|^m(1-|z_0|^2)^{1/p}}{|\lambda|^n|1-\overline{z_0}z_n|^{2/p}} + \sum_{k=n+1}^{\infty} \lambda^{-k} \frac{z_k^m f(z_k)(1-|z_0|^2)^{1/p}}{(1-\overline{z_0}z_k)^{2/p}} \\ &\quad= I + II + III. \end{aligned}$$

As in Theorem 8, choosing m to satisfy (1.15) guarantees that $II > |III|$ so that

$$\left|L_\lambda(z^m fg)\right| \ge |z_0|^m(1-|z_0|^2)^{-1/p}.$$

Since $\|z^m fg\|_{H^p} \le M$ we see that

$$\|L_\lambda\|_{H_m} \ge \frac{|z_0|^m}{M(1-|z_0|^2)^{1/p}} \ge \frac{c(p)}{M}\|ev_{z_0}\|_{H_m}.$$

Using the univalence of φ we have

$$\|C_{\varphi_n}\|_{e,H^2}^2 = \limsup_{|w|\to 1} \frac{1-|\varphi_n^{-1}(w)|}{1-|w|}$$

and, given $0 < |\lambda| < \rho' < \rho \equiv r_{e,H^p}(C_\varphi)$, we may find N_0 so that whenever $n \ge N_0$,

$$\|C_{\varphi_n}\|_{e,H^2}^{2/p} > (\rho')^n.$$

Thus for any $K \ge N_0$ we may find w in U with $|w|$ sufficiently close to 1 that

(i) $\left(\frac{1-|w|}{1-|\varphi_K(w)|}\right)^{1/p} \ge (\rho')^K$,
(ii) $|\varphi_K(w)| \ge 1/2$, and
(iii) $\frac{\|ev_{\varphi_K(w)}\|_{H_m}}{\|ev_w\|_{H_m}} \ge \frac{1}{2^m c(p)} \frac{\|ev_{\varphi_K(w)}\|_{H^p}}{\|ev_w\|_{H^p}} \approx \frac{1}{2^m c(p)}\left(\frac{1-|w|}{1-|\varphi_K(w)|}\right)^{1/p}$,

where we have used Proposition 12 in (iii).

Using (i) and (iii) we see that

$$\frac{\|ev_{\varphi_K(w)}\|_{H_m}}{\|ev_w\|_{H_m}} \ge c(p,m)(\rho')^K.$$

Choosing our iteration sequence $\{z_k\}_{-K}^{\infty}$ by setting $z_{-K} = w$ we obtain

$$\begin{aligned}\frac{\|(C_m^* - \lambda)L_\lambda\|_{H_m}}{\|L_\lambda\|_{H_m}} &\le \frac{|\lambda|^{K+1}\|ev_{z_{-K}}\|_{H_m}}{\|ev_{z_0}\|_{H_m}} \cdot M \cdot c(p) \\ &\le |\lambda| \cdot M \cdot c(m,p)\Big(\frac{|\lambda|}{\rho'}\Big)^K.\end{aligned}$$

Since m is fixed, this tends to 0 as $K \to \infty$, and $C_m^* - \lambda$ is not bounded below. ∎

We now return to the proof of the first theorem of this section.

Proof of Theorem 10: Using equation (1.8) in place of equation (1.7) and proceeding exactly as in the proof of Theorem 9, and since BMOA is continuously contained in each Hardy space, we arrive at

$$\sigma_{H^p}(C_\varphi) \subseteq \sigma_{H^1}(C_\varphi) \cup \sigma_{BMOA}(C_\varphi)$$

for all $1 \le p < \infty$. Theorem 13 now gives

$$\{\lambda \in C : |\lambda| \le r_{e,H^p}(C_\varphi)\} \subseteq \{\lambda \in C : |\lambda| \le r_{e,H^1}(C_\varphi)\} \cup \sigma_{BMOA}(C_\varphi)$$

for all $1 \le p < \infty$. The Bourdon–Shapiro essential spectral radius formula applies to give

$$r_{e,H^p}(C_\varphi) \uparrow 1, \quad \text{as } p \uparrow \infty,$$

and the desired result follows exactly as in the proof of Theorem 9. ∎

5. Examples and open questions

In this section we give some examples and discuss a conjecture. To begin with, we observe that when φ: $U \to U$ is univalent, fixes a point of U, and is not an automorphism, then $r_{e,A^2}(C_\varphi)$ (as well as $r_{e,H^2}(C_\varphi)$) is strictly less than 1. Indeed

$$\begin{aligned}\limsup_{|w|\to 1^-}\Big(\frac{1-|\varphi^{-1}(w)|}{1-|w|}\Big)^2 &= \Big[\liminf_{|w|\to 1^-}\Big(\frac{1-|w|}{1-|\varphi^{-1}(w)|}\Big)^2\Big]^{-1} \\ &= \Big[\liminf_{|w|\to 1^-}\Big(\frac{1-|\varphi(w)|}{1-|w|}\Big)^2\Big]^{-1}.\end{aligned}$$

Without loss of generality assume that 0 is the fixed point in U. Since

$$\liminf_{|w|\to 1^-}\frac{1-|\varphi(w)|}{1-|w|}$$

is the infimum of the angular derivative of φ on ∂U, by Julia's Lemma it must be strictly greater than 1 (see Lemma 7.33 of [9]) if φ is not a rotation. Then using equation (1.5) we have

$$\begin{aligned} r_{e,A^2}(C_\varphi) &= \lim_{n\to\infty} \|C_{\varphi_n}\|_{e,A^2}^{1/n} \\ &= \inf_{n\in\mathbb{N}} \|C_{\varphi_n}\|_{e,A^2}^{1/n} \\ &\le \|C_\varphi\|_{e,A^2} \\ &= \Big[\inf_{n\in\mathbb{N}} \{|\varphi'(\zeta)| : \zeta \in \partial U\}\Big]^{-1} \\ &< 1. \end{aligned}$$

A similar argument gives the same conclusion for $r_{e,H^2}(C_\varphi)$. See also [4], Proposition 3.3 where the same result is obtained under more general hypotheses on φ.

We now let $0 < r < 1$ and consider

$$\varphi(z) = \frac{rz}{1-(1-r)z}.$$

This function is univalent, fixes zero (and one), and is not an automorphism. Hence, it satisfies the hypotheses for Theorems 9 and 10. Its nth iterate is given by

$$\varphi_n(z) = \frac{r^n z}{1-(1-r^n)z}.$$

For each non-negative integer n, the angular derivative of φ_n at 1 is given by

$$(\varphi_n)'(1) = 1/r^n,$$

while at every other $\zeta \in \partial U$ the angular derivative is infinite. Thus,

$$\limsup_{|w|\to 1^-} \left(\frac{\log|\varphi^{-1}(w)|}{\log|w|}\right)^2 = \left(\varphi'(1)\right)^{-2} = r^2.$$

In this equation, φ can be replaced by φ_n. From equation (1.5) we deduce that

$$\|C_{\varphi_n}\|_{e,A^2}^2 = (r^n)^2,$$

and hence

$$r_{e,A^2}(C_\varphi) = r.$$

Theorems 5, 8, and 9 now give

$$r_{e,A^p}(C_\varphi) = r^{2/p}, \quad p > 1,$$
$$r_{e,A^1}(C_\varphi) \le r^2$$

and

$$\sigma_{A^p}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \le r_{e,A^p}(C_\varphi)\} \cup \{(\varphi'(0))^n\}_{n=0}^{\infty},$$

for each $1 \le p < \infty$. Thus

$$\{\lambda \in \mathbb{C} : r_{e,A^1}(C_\varphi) \le |\lambda| \le 1\} \subseteq \sigma_{\mathcal{B}}(C_\varphi)$$

where we know the inner radius of this annulus is at most r^2. For the same class of examples, we get

$$r_{e,H^p}(C_\varphi) = r^{1/p}, \quad p > 1,$$
$$r_{e,H^1}(C_\varphi) \le r$$

and

$$\sigma_{H^p}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \le r_{e,H^p}(C_\varphi)\} \cup \{(\varphi'(0))^n\}_{n=0}^{\infty},$$

for each $1 \le p < \infty$. Thus

$$\{\lambda \in \mathbb{C} : r_{e,H^1}(C_\varphi) \le |\lambda| \le 1\} \subseteq \sigma_{BMOA}(C_\varphi)$$

where the inner radius is at most r. We can choose r as close to zero as we wish. This shows that the annuli of Theorems 9 and 10 can be very "fat".

As previously observed when $\varphi(0) = 0$, $\sigma_{\mathcal{B}}(C_\varphi)$ and $\sigma_{BMOA}(C_\varphi)$ must each contain $(\varphi'(0))^n$, $n = 0, 1, 2, \ldots$. If the essential spectral radius of C_φ on A^2 (or H^2) is not zero, none of these points (except for $(\varphi'(0))^0 = 1$) can be eigenvalues for C_φ. The reason is that if $0 \ne (\varphi'(0))^n$ is an eigenvalue for C_φ for some $n \ge 1$, then the eigenspace is one-dimensional and spanned by the kth power of the Koenig's function F for φ (i.e., the unique analytic F on U with $F \circ \varphi = \varphi'(0)F$ and $F'(0) = 1$). But if $F^k \in \mathcal{B}$ (or $F^k \in BMOA$), then $F \in A^p$ for all $p < \infty$ ($F \in H^p$ for all $p < \infty$) which forces $\sigma_{A^p}(C_\varphi) = 0$ ($\sigma_{H^p}(C_\varphi) = 0$); see Theorem 1.2 in [15] and Theorem 4.4 in [4].

We end with a conjecture. We point out that a consequence of our work is that a nonzero value of $r_{e,A^p}(C_\varphi)$ for any $1 < p < \infty$ is equivalent to $r_{e,A^p}(C_\varphi) \ne 0$ for every value of $1 < p < \infty$ and that the same holds true for the Hardy spaces. For univalent φ, $r_{e,A^2}(C_\varphi) \ne 0$ if and only if $r_{e,H^2}(C_\varphi) \ne 0$.

Conjecture: *Suppose that φ is univalent, is not an automorphism, and fixes a point in U. If $r_{e,H^2}(C_\varphi) \ne 0$ then the spectrum of C_φ on the Bloch space or BMOA is the closed unit disk:*

$$\sigma_{BMOA}(C_\varphi) = \sigma_{\mathcal{B}}(C_\varphi) = \overline{U}.$$

Without loss of generality we may take the fixed point to be 0. Norm considerations show that $\sigma_{\mathcal{B}}(C_\varphi)$ (respectively $\sigma_{BMOA}(C_\varphi)$) is contained in the closed unit disk, so the issue is to show that

$$\{\lambda : |\lambda| < r_{e,A^1}(C_\varphi)\}$$

(respectively $\{\lambda : |\lambda| < r_{e,H^1}(C_\varphi)\}$) is in the spectrum. This would follow if the arguments of Theorems 9 and 10 could be extended to the range $0 < p < 1$, however the impediments to pursuing this approach seem formidible. A quasi-Banach space version of Theorem 2 in [18] and $p < 1$ versions of Theorems 5 and 8 (and their Hardy space analogues) would be needed. It may be more tractable to proceed by adapting the proof of Theorem 8 directly to $\mathcal{B}$ and $BMOA$. Indeed, with this approach it might be possible to omit the univalence hypothesis on φ. We leave this for another time.

References

[1] J. Arazy, S. Fisher and J. Peetre, *Möbius invariant function spaces,* Journal für die reine und angewandte Mathematik **363** (1985), 110–145.

[2] S. Axler, *Bergman spaces and their operators,* in *Surveys of Some Recent Results in Operator Theory Vol. 1* (J. B. Conway and B. B Morrel, eds.), Pitman Research Notes in Mathematics **171** (1988), 1–50.

[3] J. Bergh and J. Löfström, *Interpolation Spaces,* Springer-Verlag, Berlin, 1976.

[4] P. Bourdon, *Essential angular derivatives and maximum growth of Koenigs eigenfunctions,* Journal of Functional Analysis **160** (1998), 561–580.

[5] P. Bourdon, J. Cima and A. Matheson, *Compact composition operators on BMOA,* Transactions of the American Mathematical Society **351** (1999), 2183–2196.

[6] P. Bourdon and J. Shapiro, *Mean growth of Koenigs eigenfunctions,* Journal of the American Mathematical Society **10** (1997), 299–325.

[7] A. Calderón, *Intermediate spaces and interpolation, the complex method,* Studia Mathematica **24** (1964), 113–190.

[8] C. Cowen and B. MacCluer, *Spectra of some composition operators,* Journal of Functional Analysis **125** (1994), 223–251.

[9] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions,* CRC Press, Boca Raton, 1995.

[10] M. Cwikel, M. Milman and Y. Sagher, *Complex interpolation of some quasi-Banach spaces,* Journal of Functional Analysis **65** (1986), 339–347.

[11] R. Donaway, *Norm and essential norm estimates of composition operators on Besov type spaces,* Ph.D. Dissertation, University of Virginia, 1999.

[12] H. Kamowitz, *The spectra of endomorphisms of the disc algebra,* Pacific Journal of Mathematics **46** (1973), 433–440.

[13] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space,* Transactions of the American Mathematical Society **347** (1995), 2679–2687.

[14] A. Montes-Rodríguez, *The essential norm of a compositon operator on the Bloch spaces,* Pacific Journal of Mathematics **188** (1999), 339–351.

[15] P. Poggi-Corradini, *The Hardy class of Koenigs maps,* The Michigan Mathematical Journal **44** (1997), 495–507.

[16] P. Poggi-Corradini, *The essential norm of composition operators revisited,* in *Studies on Composition Operators (Laramie, WY, 1996)*, Contemporary Mathematics, Vol. 213, American Mathematical Society, Providence, RI, 1998, pp. 167–173.

[17] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, Berlin, 1980.

[18] K. Saxe, *Compactness-like properties preserved by complex interpolation,* Arkiv för Matematik **35** (1997), 353–362.

[19] J. Shapiro, *The essential norm of a composition operator,* Annals of Mathematics **125** (1987), 375–404.

[20] W. Smith, *Composition operators between Bergman and Hardy spaces,* Transactions of the American Mathematical Society **348** (1996), 2331–2348.

[21] M. Tjani, *Compact composition operators on some Möbius invariant Banach spaces,* Ph.D. Dissertation, Michigan State University, 1996.

[22] D. Vukotić, *A sharp estimate for* A^p_α *functions in* $\mathbb{C}^n$, Proceedings of the American Mathematical Society **117** (1993), 753–756.

[23] T. Wolff, *A note on interpolation spaces,* in *Harmonic Analysis (Minneapolis, MN, 1981)*, Lecture Notes in Mathematics **908**, Springer-Verlag, Berlin, 1982, pp. 199–204.

[24] L. Zheng, *Essential norm and spectrum of composition operators on* H^∞, preprint.

[25] K. Zhu, *Operator Theory in Function Spaces,* Marcel Dekker, New York, 1990.

[27] K. Zhu and T. MacGregor, *Coefficient multipliers between Bergman and Hardy spaces,* Mathematika **42** (1995), 413–426.